Differential Topology April 2026

End-Semester Exam 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{R}$ = real numbers, S^n = the unit sphere in $\mathbb{R}^{n+1}$.

(c) There are a total of **110** points in this paper. You will be awarded a maximum of **100** points.

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1. [10 + 10 + 10 + 10 = 40 Points] Let X be a k-manifold in $\mathbb{R}^N$.
   (i) Define the tangent bundle $T(X)$ as a subset of $\mathbb{R}^{2N}$ and verify that it is a $2k$-submanifold.
   (ii) Verify that there is a natural submersion $\pi\colon T(X) \to X$ which admits local sections, i.e., any $x \in X$ admits an open neighbourhood $V \subseteq X$ and a smooth map $\sigma\colon V \to T(X)$ such that $\pi\sigma(p) = p$ for all $p \in V$.
   (iii) For any $(x, w) \in M := T(X)$, give a canonical isomorphism
   $$T_{(x,w)}M \cong T_xX \oplus T_xX.$$
   (iv) Consider the smooth map $f\colon T(X) \to \mathbb{R}$ given by $f(x, w) = \|w\|^2$ where $\|-\|$ is the norm in $\mathbb{R}^N$. Determine the critical points of f. (Assume $k \geq 1$.)

2. [12 Points] Let S be a boundaryless k-manifold and suppose $f\colon S \to \mathbb{R}^n$ is a smooth map transversal to the hyperplane $\mathbb{R}^{n-1}$ given by $x_n = 0$. Prove that $f^{-1}\mathbb{H}^n$ is a k-manifold with boundary $f^{-1}(\mathbb{R}^{n-1})$ where $\mathbb{H}^n$ is the upper half-space given by $x_n \geq 0$.

3. [12 Points] Let X, Y be submanifolds of $\mathbb{R}^N$ with Y boundaryless. Show that for almost all $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversally everywhere.

4. [6 + 6 + 18 = 30 Points] Let $f\colon X \to Y$ be a smooth map and $Z \subset Y$ a submanifold where X, Y, Z are all boundaryless.
   (i) List all the additional conditions that are needed to define the mod 2 intersection number $I_2(f, Z)$.
   (ii) Under the conditions of (i), define $I_2(f, Z)$ without checking its well-definedness.
   (iii) For each of the conditions listed in (i), give an example showing that $I_2(f, Z)$ is not well-defined even if only that condition is relaxed.

5. [16 Points] Let X be a connected compact boundaryless $(n-1)$-manifold in $\mathbb{R}^n$. Prove that for any point z in the inside of X, there exists a ray starting at z, say $\gamma_v(t) = \{z + tv \mid t > 0, v \in S^{n-1}\}$ such that $\gamma_v(t)$ has a nonempty transversal intersection with X.
(You may assume the Jordan Brouwer separation theorem proved in class and the description of the inside and the outside regions of X in terms of winding numbers).